ETMAG

CORONALECTURE 6

Derivatives – cont.

April 27, 12:15

Example.

Just for fun, let us verify another formula for the derivative of a well-known function. Last week we verified the formula for (arc sin x)' using the theorem about the derivative of the inverse function.

Now we will verify that (sin x)' = cos x, straight from the definition.

Proof. (of (sin x)'=cos x)

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = (\textit{trigonometry})$$

$$\lim_{h\to 0} \frac{\cos x \sin h + \sin x \cos h - \sin x}{h} =$$

$$\lim_{h\to 0} \frac{\cos x \sin h - \sin x\,(1-\cos h)}{h} = (\textit{using } \lim_{h\to 0} \frac{\sin h}{h} = 1)$$

$$\cos x - \lim_{h\to 0} \frac{\sin x\,(1-\cos h)}{h} =$$

$$\cos x - \lim_{h\to 0} \frac{\sin x\,(1-\cos h)(1+\cos h)}{h(1+\cos h)} =$$

$$\cos x - \lim_{h\to 0} \frac{\sin x\,(1-\cos^2 h)}{h(1+\cos h)} =$$

$$\cos x - \sin x \cdot \lim_{h\to 0} \frac{(\sin^2 h)}{h(1+\cos h)} =$$

$$\cos x - \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{1+\cos h}}_{0} =$$

$$\cos x - (\sin x) \cdot 1 \cdot \frac{0}{2} = \cos x$$

Remark.

Finding the derivative of the function x^x (or, more general, $f(x)^{g(x)}$) is a very popular trap for students. Some differentiate it as a power function, some as an exponential function while in fact it is neither. The trick you need here is to represent this function differently, combining *ln x* function and the exponential function. Clearly one can write $x = e^{\ln x}$. This means that $x^x = \left(e^{\ln x}\right)^x = e^{x\ln x}$. Now, this is an exponential function, with the exponent x*ln* x. Using the chain rule we get

$\left(e^{x\ln x}\right)' = e^{x\ln x}\,(x\ln x)' = e^{x\ln x}(1 + \ln x) = x^x(1 + \ln x)$.

The same trick can be applied in the general case, $f(x)^{g(x)} = e^{g(x)\ln f(x)}$. Assuming, of course, that $f(x)>0$.

Example.

Find all tangent lines for f(x) = $2x^3 + 3x^2 - 11x + 6$, parallel to the line y=x.

Solution. The slope of the line y=x is 1 so me must look for points where $f'(x)$ = 1. Since $f'(x) = 6x^2 + 6x$ −11 we get

$6x^2 + 6x - 11 = 1$ or, equivalently, $x^2 + x - 2 = 0$. This yields $x_1 = 1$ and $x_2 = -2$.

The equation of the tangent at x_0 is y = $f'(x_0)(x - x_0) + f(x_0)$ and the derivative in both cases is 1, hence we get two lines,

$y = 1(x - 1) + 0 = x - 1$ at at $x_1 = 1$ and

$y = 1(x + 2) + 24 = x + 26$ at at $x_2 = -2$.

Theorem. (de l'Hospital Rule, indeterminate limits)

Let $c \in (a;b)$ and denote $D = (a;b) \setminus \{c\}$. Let f and g be functions differentiable on D such that $g(x) \neq 0$ and $g'(x) \neq 0$. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ and there exists $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ then the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and is equal to L.

Fact.

A suitably modified theorem is true for $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$

Warning 1.

Remember that $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ (or $\pm\infty$) is part of the assumption of the theorem. It does not work in other cases.

Example. If you put *f*(x)=x and *g*(x)=x+5 you get $\lim_{x\to 0} \frac{x}{x+5} = 0$.

But if you use, illegally, de l'Hospital rule you get $\lim_{x\to 0} \frac{1}{1} = 1$.

Why using de l'Hospital rule is illegal? Because the limit of g(x) is not zero and you may only apply de l'Hospital rule to *indeterminate* limits, i.e. limits of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. It would work, though, at plus or minus infinity.

Warning 2.

In de l'Hospital rule we divide the derivative of the numerator by that of the denominator.

Some students are confused by this they because they remember that $\frac{f'(x)}{g'(x)}$ is NOT the derivative of $\frac{f(x)}{g(x)}$. They are both right and wrong. They are right in that this IS NOT the way to differentiate the quotient of f(x) and g(x). And they are wrong because nobody is asking us to differentiate the quotient. We are invited to investigate the limit of the quotient of derivatives, NOT the limit of the derivative of the quotient.

Comment.

By simple algebraic tricks de l'Hostpital rule can be made to work for other indeterminate limits:

For example, in the case of $\infty - \infty$ - type limit, i.e. $\lim_{x\to c}(f-g)(x)$ where $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$ we can put

$f(x) - g(x) = \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$ which turns $\infty - \infty$ into $\frac{0}{0}$.

For example calculate

$\lim_{x\to\infty}\left(\sqrt{2x} - \sqrt{x}\right) = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2x}}}{\frac{1}{\sqrt{2\,x^2}}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x}-\sqrt{x}}{x\sqrt{2}}}{\frac{1}{x\sqrt{2}}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x}-\sqrt{x}}{x}}{\frac{1}{x}}$ good luck with this. On the other hand the limit in question $\lim_{x\to\infty}\left(\sqrt{2x} - \sqrt{x}\right) = \lim_{x\to\infty}\sqrt{x}\left(\sqrt{2} - 1\right) = \infty$ without de l'Hostpital rule. The lesson to learn – do not use the rule unless it is unavoidable.

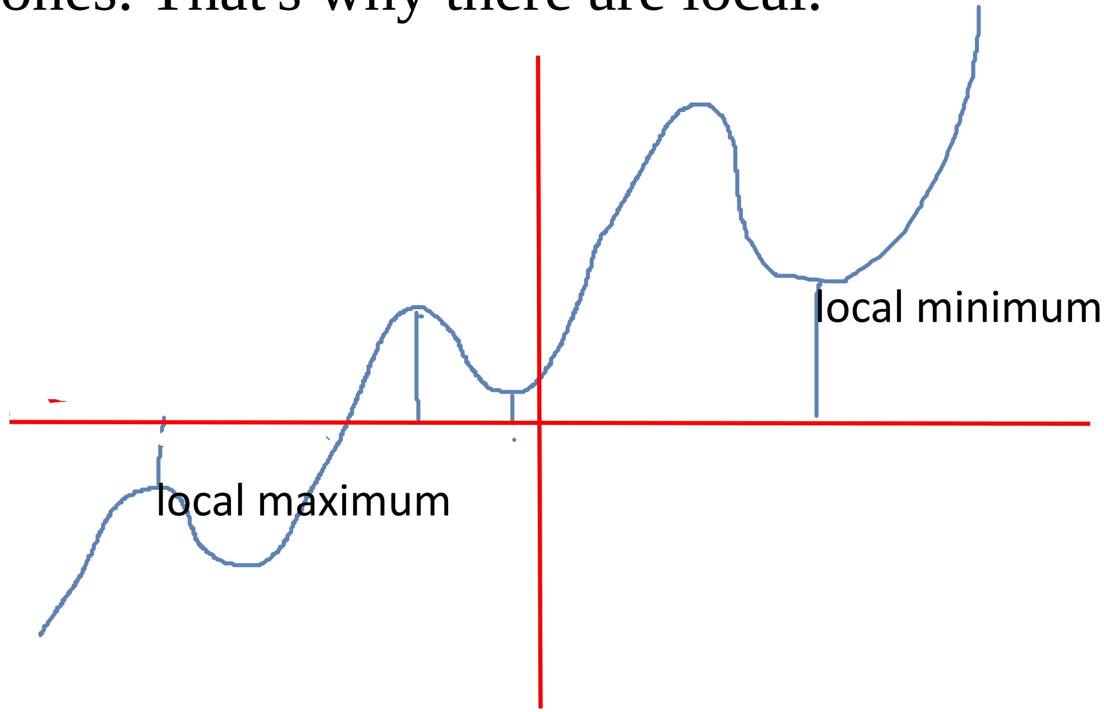
Example.

Calculate $\lim\limits_{x \to \infty} \frac{x^4}{e^x}$. This is an indeterminate limit of the type $\frac{\infty}{\infty}$. It is legal to try to apply de l'Hospital law (it is not guaranteed to work). In this case $\lim\limits_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to \infty} \frac{4x^3}{e^x}$ which is also an $\frac{\infty}{\infty}$ - type limit. Doing this again, and again and again we get $\lim\limits_{x \to \infty} \frac{24}{e^x}$ which is NOT an indeterminate expression any more so we do NOT differentiate again but we say that $\frac{c}{\infty}$ - type limit is zero.

We switch to the "old" presentation now.

We skip the part about geometric and physical application and approximations and move straight to slide 47 on extreme values of functions.

Some locally minimal values can be above some locally maximal ones. That's why there are local.



If a function f (x) has a local extremum at a point a *such that some open interval containing* a *is a subset of the domain of f*

interior point a of its domain and f '(a) *exists then f* '(a) *= 0.*

Proof. (of necessary condition for local extremum)

Without loss of generality we may assume that *f* has a local minimum at *a* on an open interval (*a-d;a+d*). Then

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \geq 0 \text{ because h > 0}$$

and $f(a+h) - f(a) \geq 0$ for small h, and

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \leq 0 \text{ because h<0}$$

and $f(a+h) - f(a) \geq 0$ for small h.

Since *f* is differentiable at *a* we get $f'_+(a) = f'_-(a) = f'(a)$, hence $f'(a) = 0$

Corollary 1. (of the mean value theorem)
If $f'(x) = 0$ on an open interval (a;b) then f is constant **on the interval**.

Proof. (by contradiction)

Suppose f is not constant on (a;b). Then so for some c,d $\in$ (a;b) $f(c) \neq f(d)$. From the MVT applied to f on (c;d), there exists $x_0 \in$ (c;d) such that $f'(x_0) = \frac{f(d)-f(c)}{d-c} \neq 0$ because $f(c) \neq f(d)$. QED

Corollary 2.
If $f(x)$ is continuous on [a; b] and differentiable on (a; b) and $f'(x) > 0$ for ~~any~~ **every** $x \in (a; b)$ then the function $y = f(x)$ is increasing **on the interval**.

Comprehension. Prove Corollary 2.

Remark. (on approximation)
The Mean Value Theorem says that (under some conditions), given two points x_0 and x there exists a point $c \in (x_0;x)$ such that

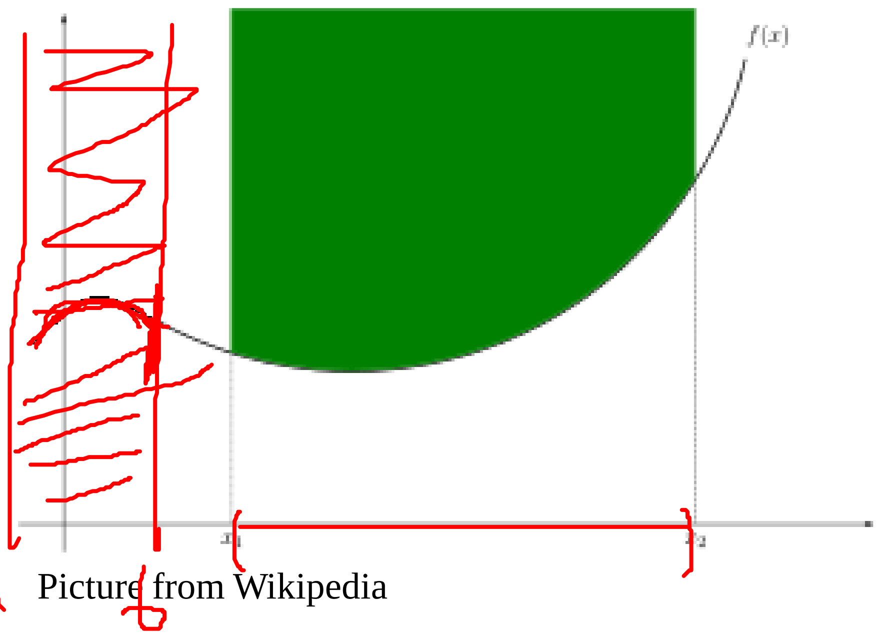
$f'(c) = \frac{f(x)-f(x_0)}{x-x_0}$ which means $f(x) - f(x_0) = f'(c)(x - x_0)$
or $f(x) = f(x_0) + f'(c)(x - x_0)$

If we replace $f'(c)$ with $f'(x_0)$ we get an *approximation* of $f(x)$. This strategy can be refined so that one can get better and better approximation of $f(x)$ using derivatives of higher and higher order. Eventually it leads to the Taylor series for a function which you will study in the second semester.

Comment. (on convexity)

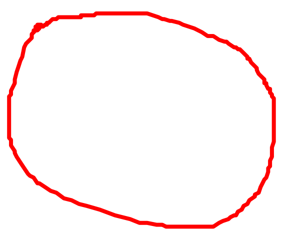
A set of points S is said to be *convex* iff for every two points $a,b \in$ S the segment joining a and b is contained in S.

A continues function is said to be *convex* on an interval (p;q) iff the area above the graph of the function is a convex set.



Picture from Wikipedia

not convex



convex